7. Krein, M. G., On some cases of the effective determination of the density of an inhomogeneous string by its spectral function. Dokl. Akad. Nauk SSSR, Vol. 93, №4, 1953.
8. Eisner, E., Inverse design for flexural vibrators. J. Acoust. Soc. Amer., Vol. 40, №4, 1966.
9. Niordson, F. J., A method for solving inverse eigenvalue problems. Recent Progress in Applied Mechanics. The Folke Odquist volume, pp. 373-382, Stockholm, 1967.
10. Niordson, F. J., Inverse problem of natural frequencies of elastic plates. III All-Union Congress on Theoretical and Applied Mechanics. Abstract Report, pp. 230-231, Moscow, 1968.
11. Akhiezer, N. and Krein, M., On Some Questions of the Theory of Moments. Khar'kov, GONTI Ukr., 1938.
12. Butkovskii, A G., Theory of Optimal Control of Systems with Distributed Parameters. Moscow, "Nauka", 1965.

Translated by M. D. F.

ON THE DENSITY OF EIGENVALUES IN PROBLEMS OF STABILITY OF THIN ELASTIC SHELLS

PMM Vol. 35, №2, 1971, pp. 364-368

N. N. BENDICH and V. M. KORNEV
(Novosibirsk)

(Received July 15, 1968)

Asymptotic estimates are determined for the density of eigenvalues. The existence of points of concentration of the eigennumbers is established. Results for the natural frequencies of shells and the eigenvalues in stability problems are compared. Conditions are written down for the solvability of the linear equation describing the stability in the presence of small perturbations.

1. The stability equation for given stress resultants in the middle surface of a shell whose radii of curvature are almost constant is

$$D\Delta\Delta\Delta\Delta\Phi_{nm} + Eh\Delta_k\Delta_k\Phi_{nm} + \lambda_{nm}\Delta\Delta(\alpha_1\Phi_{,xx} + \alpha_2\Phi_{,yy}) = 0 \qquad (1.1)$$

$$\alpha_1\lambda_{nm} = N_1, \quad \alpha_2\lambda_{nm} = N_2 \quad (0 \leqslant \alpha_1, \alpha_2 \leqslant 1), \qquad w = \Delta\Delta\Phi$$

$$\Delta\Phi = \Phi_{,xx} + \Phi_{,yy}, \quad \Delta_k\Phi = R_2^{-1}\Phi_{,xx} + R_1^{-1}\Phi_{,yy}, \quad \varphi = Eh\Delta_k\Phi$$

Here x, y are Cartesian coordinates, $w(x, y)$ is the normal shell deflection, $\varphi(x, y)$ is the stress function, Φ is the resolving function, D is the cylindrical stiffness, E, ν is the Young's modulus and Poisson's ratio, $h =$ const is the shell thickness, $R_1 \approx$ const, $R_2 \approx$ const are the radii of curvature, $-N_1$ and $-N_2$ are two constant normal compressive forces.

A rectangular shell of nonnegative Gaussian curvature, hinge-supported along the sides is considered

$$0 \leqslant x \leqslant a, \qquad 0 \leqslant y \leqslant b$$

The shell buckling mode, to the accuracy of a normalized constant, is

$$\Phi_{nm} = \sin k_n x \sin k_m y \quad (k_n = n\pi/a,\ k_m = m\pi/b,\ n, m = 1, 2, \ldots)$$

The eigennumbers are easily found

$$\lambda_{nm} = \frac{D(k_n^2 + k_m^2)^4 + Eh(R_2^{-1}k_n^2 + R_1^{-1}k_m^2)^2}{(\alpha_1 k_n^2 + \alpha_2 k_m^2)(k_n^2 + k_m^2)^2} \tag{1.2}$$

Note. The expression (1.2) written down here is the asymptotic of the eigennumbers as $n \to \infty$, $m \to \infty$ under arbitrary boundary conditions when a simple edge effect exists during stability [1–3],

$$k_n = [n + O(\varepsilon)]\,\pi / a, \quad k_n = [n + 1/2 + O(\varepsilon)]\,\pi / a, \quad k_n = [n + 1 + O(\varepsilon)]\,\pi / a$$

Here ε is a small parameter characterizing the rapidity of edge effect damping during stability. Analogous relations are also valid for k_m. It is understood that only those boundary conditions are considered in which the parameter λ_{nm} does not enter.

Let us utilize the asymptotic formula (1.2) to obtain estimates of the density of the eigenvalues. According to an idea of R. Courant [4], let us define the number of eigenvalues $A(\lambda_0)$ less than a given value λ_0, approximately as the ratio between the domain area Ω in the $k_n k_m$ plane within which is the eigenvalue $\lambda < \lambda_0$, and the area of one cell $\Delta k_n \Delta k_m$, i.e.,

$$A(\lambda_0) = \frac{1}{\Delta k_n \Delta k_m} \int_\Omega dk_n dk_m \tag{1.3}$$

Let us introduce the notation

$$k_n^2 + k_m^2 = r^2, \qquad \frac{k_m}{k_n} = \operatorname{tg}\theta, \qquad \frac{Eh^3}{R_1^2} = \lambda^*, \qquad \chi = \frac{R_1}{R_2}$$

Formula (1.2) becomes

$$\lambda = \frac{Dr^4}{r^2(\alpha_1\cos^2\theta + \alpha_2\sin^2\theta)} + \frac{\lambda^*(\chi\cos^2\theta + \sin^2\theta)^2}{h^2 r^2(\alpha_1\cos^2\theta + \alpha_2\sin^2\theta)} \tag{1.4}$$

In the new variables the relationship (1.3) is

$$A(\lambda_0) = \frac{ab}{\pi^2} \int_{\theta_1}^{\theta_2}\int_0^r r\,dr\,d\theta$$

If the inner integral is taken, and the value of r^2 from (1.4) is substituted, then we have for the average number of eigenvalues less than a given value λ_0 (the subscript on the parameter λ will henceforth be omitted throughout)

$$A(\lambda) = \frac{ab}{4\pi^2 D}\int_{\theta_1}^{\theta_2} \lambda(\alpha_1\cos^2\theta + \alpha_2\sin^2\theta)\,d\theta + \frac{ab}{4\pi^2 D}\int_{\theta_1}^{\theta_2}\Big[\lambda^2(\alpha_1\cos^2\theta + \alpha_2'\sin^2\theta)^2 - \frac{4D\lambda^*}{h^2}(\chi\cos^2\theta + \sin^2\theta)^2\Big]^{1/2} d\theta \tag{1.5}$$

Integration is over that part of the quadrant $k_n > 0$, $k_m > 0$ within which the expression in the square brackets is positive. After term-by-term differentiation of (1.5), the relationship

$$\frac{s(\lambda)}{s_0} = \frac{1}{s_0}\frac{dA(\lambda)}{d\lambda} = I_1 + I_2 \tag{1.6}$$

is obtained for the density of the eigenvalues.

The function $A(\lambda)$ is a nondecreasing function. Since the expression under the second integral can be significantly greater than the expression under the first integral, the minus sign which appears in finding r^2 from (1.4) is omitted in (1.5). The following notation is inserted into (1.6)

$$s_0 = \frac{ab}{4\pi}, \qquad I_1 = \frac{1}{\pi D}\int_{\theta_1}^{\theta_2} (\alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta)\, d\theta$$

$$I_2 = B\int_{\xi_1}^{\xi_2} \frac{[v^2 + 2v(1-v)\xi + (1-v)^2\xi^2]}{[\xi(1-\xi)(c_1-\xi)(c_2+\xi)]^{1/2}}, \qquad \sin^2\theta = \xi$$

$$\frac{\alpha_1}{\alpha_2} = v \ (\alpha_2 \neq 0), \qquad \frac{4D\lambda^*}{\lambda\alpha_2^2 h^2} = \eta^2, \qquad \frac{v - \eta\chi}{\eta(1-\chi) - (1-v)} = c_1$$

$$\frac{v + \eta\chi}{\eta(1-\chi) + (1-v)} = c_2, \qquad B = \frac{\alpha_2}{2\pi D\,[\eta^2(1-\chi)^2 - (1-v)^2]^{1/2}}$$

If $\chi > v$, the x- and y-axes must be interchanged, and then the analyzed case $0 \leqslant \chi \leqslant v$ is obtained because shells of only nonnegative Gaussian curvature are considered.

Four cases are possible for such χ :

$$c_2 > 0,\ c_1 > 1; \quad c_2 > 0,\ c_1 < 1; \quad c_2 < 0,\ c_1 > 1; \quad c_2 < 0,\ c_1 < 1$$

The first case is thus $c_2 > 0$, $c_1 > 1$. The radical in I_2 is positive for $\xi_1 = 0$, $\xi_2 = 1$, and the whole integral reduces to (see [5])

$$I_2 = 2B^* [c_1(c_2+1)]^{-1/2} \{[v^2 - 2v(1-v)c_2 + {}^1/_2(1-v)^2 c_1 c_2 (c_2+1)(c_2-1)]\times$$
$$\times K(q) - {}^1/_2(1-v^2)c_1(c_2+1)\,\mathrm{E}(q) + [2v(1-v)c_2 + {}^1/_2(1-v)^2 c_1 c_2 (1 - c_2 + c_1)(c_2+1)^2]\,\Pi(\varkappa, q)\}$$

Here $K(q)$, $\mathrm{E}(q)$, $\Pi(\varkappa, q)$ are the complete elliptic integrals of the first, second and third kinds, respectively, in Legendre form, where

$$q = \left(\frac{2\eta(v-\chi)}{(v-\eta\chi)(\eta+1)}\right)^{1/2}, \qquad \varkappa = \frac{\eta(1-\chi)+1-v}{\eta+1}, \qquad 2B^* = \frac{\alpha_2[c_1(c_2+1)]^{1/2}}{\pi D[(v-\eta\chi)(1+\eta)]^{1/2}}$$

In this case

$$I_1 = (\alpha_1 + \alpha_2)\,/\,4D$$

The remaining three cases and the special case when $\alpha_2 = 0$ are considered analogously.

2. All the cases can be combined as follows:

For $v - \eta\chi < 0$

$$s(\lambda) = 0 \tag{2.1}$$

For $v - \eta\chi > 0$ $\quad(\eta < 1)$

$$\frac{s(\lambda)}{s_0} = A_0 + \frac{\alpha_2[A_1 K(q) + A_2 \mathrm{E}(q) + A_3 \Pi(\varkappa, q)]}{\pi D[(v-\eta\chi)(1+\eta)]^{1/2}}$$

$$\varkappa = \frac{\eta(1-\chi)+1-v}{\eta+1} \quad \text{or} \quad \varkappa = \frac{2\eta(v-\chi)}{[\eta(1-\chi)+1-v](v-\eta\chi)}$$

For $v - \eta\chi > 0$ $(\eta > 1)$

$$\frac{s(\lambda)}{s_0} = B_0 + \frac{\alpha_2[B_1 K(q^{-1}) + B_2 \mathrm{E}(q^{-1}) + B_3 \Pi(-\varkappa, q)]}{\pi D[2\eta(v-\chi)]^{1/2}}$$

$$\varkappa = \frac{[\eta(1-\chi)+1-v](v-\eta\chi)}{2\eta(v-\chi)} \quad \text{or} \quad \varkappa = \frac{\eta+1}{\eta(1-\chi)+1-v}$$

$$q = [2\eta(v-\chi)/(v-\eta\chi)(\eta+1)]^{1/2}$$

Here A_0, B_0 are certain constants; A_1, A_2, A_3, B_1, B_2, B_3 depend on v, c_1, c_2.

Formulas (2.1) permit disclosure of certain properties of the density of the eigenvalues in problems of thin elastic shell stability. As follows from the formulas for $v - \eta\chi < 0$ the density of the eigenvalues is zero (see Fig. 1). Curve *1* in Fig. 1 corresponds to the existence of two poles for $\chi < 1$, Curve *2* to the existence of just the pole η_1 for $\chi \geqslant 1$ and Curve 3 to $\chi = v$.

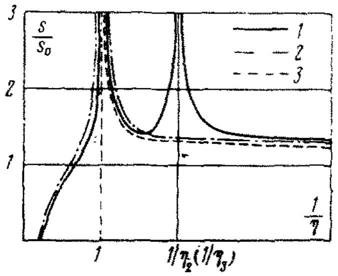

Fig. 1

For $v - \eta\chi > 0$ the density of the eigenvalues is different from zero, where as $\eta \to 0$, the function $s(\lambda)$ tends to some constant connected with the density of the eigenvalues in plate stability problems, and for $\eta = 1$ the function $s(\lambda)$ has a pole (actually, for $\eta = 1$ the argument of each of the complete elliptic integrals becomes one and the integral diverges). Let η_1 be this pole.

Since each of the coefficients in the integrals $(\alpha_1 \neq \alpha_2)$, can have a singularity, still another pole appears in the function if $v < 1$, then this pole is at the point $\eta = (1 - v) / (1 - \chi)$ (the denominator c_1 vanishes). Let it be denoted by η_2. If however $v > 1$ and $\chi < 1$, the pole is at the point $\eta = (v - 1) / (1 - \chi)$ (the denominator c_2 vanishes). Let η_3 be this pole. When $v > 1$ and $\chi \geqslant 1$, $1 / \eta < \chi / v$, and this means there is no second pole. Since $v > \chi$, then $\eta_2 < \eta_1$. Depending on the quantities v and χ, the quantity η_3 can be either greater or less than η_1.

Finally, when v coincides with χ

$$\frac{s(\lambda)}{s_0} = \frac{c_0}{(1 - \eta^2)^{1/2}} \qquad (c_0 = \text{const})$$

there exists just one pole at the point $\eta = 1$. Precisely this case, probably, is of essential value for applied problems since the distribution of the eigenvalues starts at the point of concentration. Among this class of problems are problems on the stability of a longitudinally compressed cylindrical shell and a sphere under hydrostatic pressure. As is known, these problems are solved in a classical formulation without determining the buckling mode [6].

In the special case $(\alpha_2 = 0)$, the first pole η_1 is missing, and a pole can exist at the point $\eta = 1 / (1 - \chi)$ for $\chi < 1$ or at the point $\eta = 1 / (\chi - 1)$ for $\chi > 1$. However this pole does not exist for $\chi = 1$.

When the compressive stresses are equal $(\alpha_1 = \alpha_2)$, the function $s(\lambda)$ has just one pole at the point $\eta = 1$, because the elliptic integrals of the second and third kind in the formulas vanish since their coefficients vanish. This latter case is reminiscent of the results obtained for the natural frequencies of shells in [7, 8].

3. The eigenvalue distribution can turn out to be useful in solving stability problems when there are small perturbations, for example, a small initial deflection. The linear equation describing the stability in the presence of small perturbations formally has the form

$$A\Phi - \lambda B\Phi = f(x, y) \tag{3.1}$$

Here A and B are some positive-definite operators, λ is the loading operator, $f(x, y)$ is a function characterizing the small perturbations. The boundary conditions for (3.1) are homogeneous conditions.

For certain boundary conditions let the eigenfunctions Φ_{nm} and corresponding eigenvalues λ_{nm} be known for the problem

$$A\Phi_{nm} - \lambda_{nm} B\Phi_{nm} = 0$$

where the orthogonality and normalization conditions

$$(\Phi_{ij}, B\Phi_{nm}) = \delta_{in}\delta_{jm}$$

are satisfied for the former. Here δ_{in}, δ_{jm} are Kronecker deltas. Let us represent the solution of (3.1) in the form

$$\Phi = \sum_{n,m} \alpha_{nm}\Phi_{nm}, \quad \alpha_{nm} = \frac{f_{nm}}{\lambda_{nm} - \lambda}, \quad f_{nm} = (\Phi_{nm}, f) \tag{3.2}$$

If the parameter λ equals the least eigenvalue Λ_{NM} ($\lambda = \Lambda_{NM}$), then for the solvability of (3.1)

$$(\Phi_{NM}, f) \equiv 0 \tag{3.3}$$

The subscripts N and M here denote the eigenfunctions corresponding to Λ_{NM}. Condition (3.3) is the analog of the existence condition for the solution of the inhomogeneous Helmholtz equation when the characteristic parameter agrees with an eigenvalue.

It has been established above that the distribution of the eigenvalues sometimes starts with the point of concentration, i.e. an infinity of different eigenfunctions have coincident eigenvalues.

As $\lambda \to \Lambda_{NM}$ all the $\alpha_{NM} \to \infty$, where $\alpha_{NM}/\alpha_{N_*M_*} = O(1)$, if f_{NM} are of the same order in (3.2). Therefore, the representation of a shell with a small initial imperfection in the form of a system with one degree of freedom can result in great errors [9–11] in this case.

When just one eigenfunction Φ_{NM}, corresponds to the least eigenvalue, the condition for the solvability of (3.3) is satisfied more simply, and the replacement of a system with an infinite number of degrees of freedom with a system with one degree of freedom will not result in great errors.

Another interpretation can be given to the solvability condition for (3.3). It is sufficient to recall [11] that experiments on the stability of longitudinally compressed cylindrical and spherical shells under hydrostatic pressure have huge scatter, and moreover, the magnitude of the experimental critical loading turns out, as a rule, to be considerably less than the classical loading. Loadings agreeing with the classical critical loading were obtained in none of the experiments. This is associated with the fact that

$$\alpha_{NM} \to \infty \quad \text{for} \quad \lambda \to \Lambda_{NM} \quad (f_{NM} \neq 0)$$

Precisely in these problems the distribution of the eigenvalues start with the point of concentration.

The scatter is considerably less [11] in experiments on the stability of cylindrical shells subjected to external transverse or multilateral hydrostatic pressure; there are even individual experiments in which the critical exceeds the classical loading. For these problems just two eigenfunctions correspond to the least eigenvalue, and the change in boundary conditions (of the hinged-support type) does not affect the magnitude of the critical pressure essentially since a simple edge effect exists during stability [2].

BIBLIOGRAPHY

1. Bolotin, V. V., The edge effect in the oscillations of elastic shells. PMM Vol. 24, №5, 1960.
2. Kornev, V. M., Stability of a circular cylindrical shell, loaded by external transverse pressure, taking edge effect into account. Inzh. Zh., Mekh. Tverdogo Tela, №3, 1967.
3. Vishik, M. I. and Liusternik, L. A., Solution of some problems on perturbations in the case of matrices and self-adjoint and non-self-adjoint differential equations. I. Usp. Matem. Nauk, Vol. 15, №3, 1960.
4. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 2. Moscow-Leningrad, Gostekhizdat, 1951.
5. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1963.
6. Vlasov, V. Z., Selected Works, Vol. 1. Moscow, Akad. Nauk SSSR Press, 1962.
7. Bolotin, V. V., On the density of the distribution of natural frequencies of thin elastic shells. PMM Vol. 27, №2, 1963.
8. Bolotin, V. V., Eigenvalue density in problems of elastic plate and shell vibrations. Trans. of VI All-Union Conf. on Plates and Shells Theory (Baku, 1966). Moscow, "Nauka", 1966.
9. Donnell, L. H., A new theory for the buckling of thin cylinders under axial compression and bending. Trans. ASME, Vol. 56, №10, 1934.
10. Kármán, Th. and Tsien, H. S., The buckling of thin cylindrical shells under axial compression. J. Aero. Sci., Vol. 8, №8, 1941.
11. Vol'mir, A. S., Stability of Elastic Systems. Moscow, Fizmatgiz, 1963.

Translated by M. D. F.

ON EXTREMAL STRESSES IN THE PLANE PROBLEM OF THE THEORY OF ELASTICITY

PMM Vol. 35, №2, 1971, pp. 369-375
S. A. KAS'IANIUK and T. I. TKACHUK
(Kiev)
(Received July 21, 1969)

The problem of extremal stresses in the first fundamental problem for a half-plane and a circle depending on the stress distribution on the contour is studied by using estimates for the integral operators of plane elasticity theory. S. A. Kas'ianiuk solved the problems for the half-plane and G. I. Tkachuk for the circle.

It is known from [1], p. 293 and from [2] that the stress components X_x, X_y, Y_y at the point $z = x + iy$ in the first fundamental plane problem for the half-plane $y < 0$ are defined in terms of the normal $N(t)$ and tangential $T(t)$ stresses given along the x-axis by using the equalities

$$X_x + Y_y = 4\operatorname{Re}\Phi(z) \tag{0.1}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)]$$